VARIATIONAL PROBLEM ON THREE-DIMENSIONAL SUPERSONIC FLOWS

(VARIATSIONNAIA ZADACHA O TREKHMERNYKH SVERKHZVUKOVYKH TECHENIIAKH)

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Considered is the problem of the construction of the supersonic part of a nozzle with a maximum thrust. The first exact solution, in closed form, of this problem for an axially symmetric supersonic flow was given by Shmyglevskii [1]. The end points of the nozzle generatrix were considered given. For the explicit description of the functional and auxiliary conditions, use was made of the transition from the contour of the body to the boundaries of the region of influence.

In the present work the supersonic flow in the nozzle is assumed to be spatial. The differential equations of flow are used as relations between functions. This approach to the solution of variational problems of gas dynamics was used by Guderley and Armitage [2] and by Sirazetdinov [3]. The necessary conditions for an extremum which are obtained in this formulation of the problem represent a boundary value problem for a system of nonlinear partial differential equations with conditions on the entire surface which bounds the region of influence. An analogous result was obtained, for example, in [2] in the determination of an axially symmetric nozzle of maximum thrust with arbitrary isoperimetric conditions on the walls.

Under sertain restrictions which are related only to the contour of the outlet of the nozzle, there exists a class of spatial optimum solutions in which the number of inde endent variables of the boundary value problem can be decreased. For an axially symmetric flow this was done in paper [4].

1. Formulation of the variational problem. Let u, v, and w be the projections of the velocity on the axes of a Cartesian coordinate system x, y, z. For the description of a stationary irrotational isentropic flow of a nonviscous nonheatconducting gas, with arbitrary thermodynamic properties, it is sufficient to use three equations (two projections of the vortex and the equation of continuity):

$$L_1 \equiv u_z - w_x = 0, \qquad L_2 \equiv v_x - u_y = 0$$
$$L_3 \equiv (\rho u)_x + (\rho v)_y + (\rho w)_z = 0 \tag{1.1}$$

Here, and in what follows, the subscripts x, y and z denote partial derivatives. The density ρ, the pressure P, and the sound velocity a, are known functions of the absolute value of the velocity. Hereby,

$$\frac{dp}{\rho} = a^2 \frac{d\rho}{\rho} = -u\,du - v\,dv - w\,dw \qquad (1.2)$$

For later use, we introduce two "stream functions" $\psi(x,y,z)$ and $\chi(x,y,z)$ of the spatial flow by means of Formulas

$$\rho u = \frac{D(\psi, \chi)}{D(y, z)} = \psi_y \chi_z - \psi_z \chi_y \qquad (uvw,\ xyz) \qquad (1.3)$$

Here the symbol $(uvw,\ xyz)$ indicates a cyclic transposition.

It is not difficult to verify that Equation $L_3 = 0$ of the system (1.1) is implied by the system (1.3). Hence any two equations of (1.3) permit one to construct the stream functions ψ and χ for any known flow field.

Let us consider the differential equations of the stream lines

$$\frac{dx}{\rho u} = \frac{dy}{\rho v} = \frac{dz}{\rho w} \qquad (1.4)$$

Taking (1.3) into consideration we can perform the integration of (1.4). The calculations show that along the stream lines

$$\psi = \text{const}, \qquad \chi = \text{const} \qquad (1.5)$$

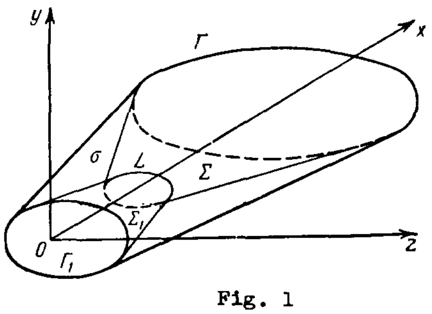

Fig. 1

Next, let us consider the variational problem.

Suppose that the parameters of the initial flow are given by the characteristic surface Σ_1. This surface (Fig.1) passes through the given contour Γ_1. Let another contour Γ be given. We shall indicate by the letter Σ an unknown closing characteristic surface which passes through Γ. The contour L is the curve in which Σ and Σ_1 intersect. Let us denote by σ the flow surface $f(x,y,z) = 0$ which passes through the contours Γ_1 and Γ. On this surface

$$u \cos nx + v \cos ny + w \cos nz = 0 \qquad (1.6)$$

Here $\mathbf{n}$ is the normal to the surface σ.

If we denote by p_0 the exterior pressure then the thrust of the nozzle in the direction x is given by the relation

$$T = \iint_\sigma (p - p_0) \cos nx \, d\sigma \qquad (1.7)$$

In supersonic flow the distribution of the pressure p on σ depends only on the region τ bounded by the surfaces Σ_1, Σ and σ.

Let us formulate the following variational problem: on the basis of a given characteristic surface Σ_1 we are to find a flow surface σ which passes through given contours Γ_1 and Γ and which yields an extremum of the functional (1.7) under the differential relation (1.6) on σ, and the differential relations (1.1) and (1.2) in the region τ.

2. Necessary conditions for an extremum. Let us denote by $c\,(x, y, z)$, $h_1\,(x, y, z)$, $h_2\,(x, y, z)$ and $h_3\,(x, y, z)$ the Lagrange multipliers. We construct Expression

$$T^\circ = \iint_\sigma [(p - p_0 + cu)\cos nx + cv\cos ny + cw\cos nz]\, d\sigma +$$

$$+ \iiint_\tau (h_1L_1 + h_2L_2 + h_3L_3)\, d\tau \qquad (2.1)$$

and require that it takes on an extremum as we vary u, v, w and x, y, z on the surface σ.

Hereby p and ρ will be functions of u, v and w, and in view of (1.2) we have

$$\delta p = -\rho u\,\delta u - \rho v\,\delta v - \rho w\,\delta w, \qquad \delta\rho = -\frac{\rho u}{a^2}\delta u - \frac{\rho v}{a^2}\delta v - \frac{\rho w}{a^2}\delta w \quad (2.2)$$

Following [2], we perform the variation of the surface and of the velocities separately. The total variation T° will be

$$\delta T^\circ = \delta T^\circ_{v=\text{const}} + \delta T^\circ_{\sigma=\text{const}}$$

In the evaluation of $\delta T^\circ_{v=\text{const}}$ one may consider three types of representation of the function $f(x, y, z) = 0$ in explicit form. One may think of $f(x, y, z) = 0$ having been solved for x; then y and z are considered as independent variables.

The quantities x_y, x_z are partial derivatives of x with resect to y and z, respectively, and are obtained from $f(x, y, z) = 0$ under the assumption that this equation determines x as a function of y and z.

Furthermore, the symbols $\partial c / \partial x$, $\partial c / \partial y$, $\partial c / \partial z$ will denote partial derivatives of the function c on the surface σ.

In the example under consideration

$$\frac{\partial c}{\partial x} = c_x, \qquad \frac{\partial c}{\partial y} = c_y + c_x x_y, \qquad \frac{\partial c}{\partial z} = c_z + c_x x_z$$

Besides that, it is clear that

$$0 = f_y + f_x x_y, \qquad 0 = f_z + f_x x_z$$

Thus, the argument x represents explicitly the surface σ in view of Equation $f(x, y, z) = 0$.

Let us evaluate $\delta T^\circ_{v=\text{const}}$ by varying the form of the surface σ, and let us set $\delta T^\circ_{v=\text{const}}$ equal to zero

$$\delta T^\circ_{v=\text{const}} = \delta \iint_\sigma [(p - p_0 + cu)\cos nx + cv\cos ny + cw\cos nz]\, d\sigma =$$

$$= \delta \iint_{\sigma_{yz}} [-(p - p_0 + cu) + cvx_y + cwx_z]\, dydz =$$

$$= \iint_{\sigma_{yz}} \left\{\left[-p_x - (cu)_x + (cv)_x x_y + (cw)_x x_z - \frac{\partial cv}{\partial y} - \frac{\partial cw}{\partial z}\right] \times \right.$$

$$\left. \times\, \delta x + \frac{\partial (cv\delta x)}{\partial y} + \frac{\partial (cw\delta x)}{\partial z}\right\} dydz = 0$$

The symbol σ_{yz} denotes here the projection of the surface σ on the plane yz. Using Green's formula. and taking into account the fact that $\delta x = \delta n \cos nx$, we obtain

$$\delta T^{\circ}_{v=\text{const}} = \iint\limits_{\sigma} - [p_x + (cu)_x + (cv)_y + (cw)_z]\, \delta n \cos^2 nx d\sigma = 0$$

The integral of this expression, which has the form of a divergence, vanishes since the boundaries of the region of integration remain fixed.

The quantity $\cos nx$ is not equal to zero in general on the surface σ. Hence, if $\delta T^{\circ}_{v=\text{const}}$ is to be zero it is necessary that we have the following relation on σ:

$$p_x + (cu)_x + (cv)_y + (cw)_z = 0$$

Since Equations (1.1) and (1.2) are valid in τ, the last displayed equation may be rewritten as

$$-\rho u u_x - \rho v u_y - \rho w u_z + \rho u \left(\frac{c}{\rho}\right)_x + \rho v \left(\frac{c}{\rho}\right)_y + \rho w \left(\frac{c}{\rho}\right)_z =$$

$$= \rho u \left(\frac{c}{\rho} - u\right)_x + \rho v \left(\frac{c}{\rho} - u\right)_y + \rho w \left(\frac{c}{\rho} - u\right)_z = 0 \qquad (2.3)$$

The characteristic system of the linear homogeneous equation (2.3) coincides with the differential equations of the stream lines (1.4). Along the stream lines, $\psi = \text{const}$ and $\chi = \text{const}$ by Formula (1.5). Thus, the general solution of the linear homogeneous equation (2.3) has the form

$$\frac{c}{\rho} - u = \Phi(\psi, \chi), \qquad \text{or} \qquad c = \rho\, [u + \Phi(\psi, \chi)]$$

Here $\Phi(\psi, \chi)$ is an arbitrary function of ψ and χ.

On the flow surface the quantities ψ and χ are related. Suppose this relation is given by $\psi = \psi(\chi)$ on the surface σ. Then the variable multiplier c is given on the surface σ by Formula

$$c = \rho\, \{u + \Phi\, [\psi(\chi), \chi]\} \qquad (2.4)$$

Now we obtain an expression for $\delta T^{\circ}_{\sigma=\text{const}}$, and we set it equal to zero

$$\delta T^{\circ}_{\sigma=\text{const}} = \iint\limits_{\sigma} [(\delta p + c\delta u) \cos nx + c\delta v \cos ny + c\delta w \cos nz]\, d\sigma +$$

$$+ \iiint\limits_{\tau} \{h_1\, [(\delta u)_z - (\delta w)_x] + h_2\, [(\delta v)_x - (\delta u)_y] + h_3\, [(\delta \rho u)_x + (\delta \rho v)_y +$$

$$+ (\delta \rho w)_z]\}\, d\tau = 0 \qquad (2.5)$$

Let us denote by $\mathbf{k}$ the normal to the characteristic surface of the first family Σ. On Σ we have

$$u \cos kx + v \cos ky + w \cos kz = a \qquad (2.6)$$

Making use of the Gauss-Ostrogradskii formula, we transform the second integral of Expression (2.5) with the aid of integration by parts. Hereby, we recall that the variations of the functions vanish on the given characteristic surface Σ. In view of (2.2) we now have

$$\delta T^{\circ}_{\sigma=\mathrm{const}} = \iint_G (U_1\delta u + V_1\delta v + W_1\delta w)\, d\sigma + \iint_\Sigma (U_2\delta u + V_2\delta v + W_2\delta w)\, d\Sigma + \\ + \iiint_\tau (U_3\delta u + V_3\delta v + W_3\delta w)\, d\tau = 0 \qquad (2.7)$$

Equating to zero the expression δu, δv and δw, we determine the Lagrange multipliers on the surfaces σ, Σ and in the volume τ.

From the first integral of Formula (2.7), considering (1.6), we obtain the following conditions which must be satisfied on the surface σ:

$$\begin{aligned} U_1 &\equiv (-\rho u + c + \rho h_3)\cos nx - h_2 \cos ny + h_1 \cos nz = 0 \\ V_1 &\equiv (-\rho v + h_2)\cos nx + (c + \rho h_3)\cos ny = 0 \\ W_1 &\equiv (-\rho w - h_1)\cos nx + (c + \rho h_3)\cos nz = 0 \end{aligned} \qquad (2.8)$$

Let us introduce the notation

$$\lambda_1 = w + \frac{h_1}{\rho}, \qquad \lambda_2 = v - \frac{h_2}{\rho}, \qquad \lambda_3 = u + \Phi + h_3 \qquad (2.9)$$

In view of (1.6), (2.4) and (2.9), Equations (2.8) may now be rewritten as

$$\begin{aligned} \lambda_1 \cos nz + \lambda_2 \cos ny + \lambda_3 \cos nx &= 0 \\ -\lambda_2 \cos nx + \lambda_3 \cos ny &= 0 \\ -\lambda_1 \cos nx + \lambda_3 \cos nz &= 0 \end{aligned} \qquad (2.10)$$

The determinant Δ of the homogeneous system of equations (2.10), which determine the quantities λ_1, λ_2 and λ_3, is equal to $-\cos nx$. On the surface σ, the quantity $\cos nx$ is, in general, not qual zero. Hence, $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Recalling (2.9), we find that on the surface of the nozzle we have Equations

$$h_1 = -\rho w, \qquad h_2 = \rho v, \qquad h_3 = -\{u + \Phi\,[\psi(\chi), \chi]\} \qquad (2.11)$$

From the second integral of (2.7) and from (2.6) we obtain conditions which must be satisfied on the characteristic surface Σ

$$\begin{aligned} U_2 &\equiv h_3\rho \cos kx - h_2 \cos ky + h_1 \cos kz - h_3\rho u / a = 0 \\ V_2 &\equiv h_2 \cos kx + h_3\rho \cos ky - h_3\rho v / a = 0 \\ W_2 &\equiv - - h_1 \cos kx + h_3\rho \cos kz - h_3 \rho w / a = 0 \end{aligned}$$

This is a homogeneous system. Its determinant is zero. Hence, on Σ it is sufficient that the following two conditions be fulfilled:

$$\begin{aligned} h_2 \cos kx + h_3\rho \cos ky - h_3\rho v / a &= 0 \\ -h_1 \cos kx + h_3\rho \cos kz - h_3 \rho w / a &= 0 \end{aligned} \qquad (2.12)$$

Finally, from the third integral of Expression (2.7) we obtain the conditions which must be satisfied in the volume τ

$$U_3 \equiv h_{1z} - h_{2y} + \rho\left(1 - \frac{u^2}{a^2}\right)h_{3x} - \rho\,\frac{uv}{a^2}\,h_{3y} - \rho\,\frac{uw}{a^2}\,h_{3z} = 0$$

$$V_3 \equiv h_{2x} - \rho\,\frac{uv}{a^2}\,h_{3x} + \rho\left(1 - \frac{v^2}{a^2}\right)h_{3y} - \rho\,\frac{vw}{a^2}\,h_{3z} = 0 \qquad (2.13)$$

$$W_3 \equiv -h_{1x} - \rho\,\frac{uw}{a^2}\,h_{3x} - \rho\,\frac{vw}{a^2}\,h_{3y} + \rho\left(1 - \frac{w^2}{a^2}\right)h_{3z} = 0$$

Analysis shows that the system (2.13) for supersonic flow is of the hyperbolic type, and the characteristic directions coincide with the characteristic directions of the equations of gas dynamics (1.1) and (1.2).

Thus, the variational problem of the determination of the nozzle surface σ possessing the maximum thrust and which passes through the contours Γ_1, and Γ, has been reduced to a boundary value problem for a partial differential equation.

Indeed, let σ be some surface which is stretched over Γ_1 and Γ. On the basis of a given initial flow on Σ_1, and on the basis of the surface σ, we determine, by means of the solution of the system (1.1),(1.2), the flow in the volume τ, and also the characteristic surface of the first family Σ. Furthermore, with a given flow field and for a certain function $\Phi[\psi(\chi), \chi]$ we compute the values of h_1, h_2, h_3 on the surface σ by means of Formula (2.11). After that, by solving Cauchy's problem for Equations (2.13) in the volume τ, we find the values of h_1, h_2, h_3 on Σ. If, in addition, the condition (2.12) is satisfied on Σ, then the flow surface σ will yield the solution of the variational problem.

3. Decreasing the number of independent variables in the boundary value problem $\Phi[\psi(\chi), \chi] = \mathrm{const}$. Let us project the characteristic surface of the first family Σ, which is stretched over the contours Γ and L, upon the plane yz. In Fig.2 the bounded doubly-connected region D represents the projection of Σ upon the plane yz. The contours Γ and L are projected on γ and l, respectively. Now we rewrite the conditions which are satisfied on the surface Σ, whose equation is written in the form $\varphi(y, z) - x = 0$.

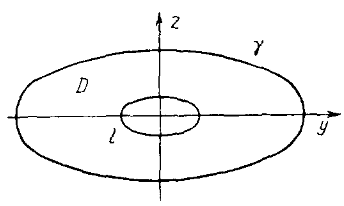

Fig. 2

We shall use the notation

$$A = a\sqrt{1 + \varphi_y^{\,2} + \varphi_z^{\,2}} \qquad (3.1)$$

Then the two conditions of extremality (2.12) take the form

$$\rho v\,\frac{A}{a^2} = -\frac{h_2}{h_3} + \rho\varphi_y, \qquad \rho w\,\frac{A}{a^2} = \frac{h_1}{h_3} + \rho\varphi_z \qquad (3.2)$$

and the condition of directions (2.6) becomes

$$-u + v\varphi_y + w\varphi_z = A \qquad (3.3)$$

The surface Σ is the characteristic surface of the system (2.13). The condition of coincidence on this surface has the form

$$\frac{\partial h_1}{\partial z} - \frac{\partial h_2}{\partial y} + \rho \frac{A}{a^2}\left(v \frac{\partial h_3}{\partial y} + w \frac{\partial h_3}{\partial z}\right) - \rho\varphi_y \frac{\partial h_3}{\partial y} - \rho\varphi_z \frac{\partial h_3}{\partial z} = 0$$

Taking into account (3.2), we may rewrite the condition of coincidence on Σ as

$$\frac{\partial h_1 h_3}{\partial z} - \frac{\partial h_2 h_3}{\partial y} = 0 \tag{3.4}$$

Let us set $\Phi\,[\psi(\chi), \chi] = c_1 = \text{const}$ on the surface σ. Next, we consider the expressions for the Lagrange multipliers

$$h_1 = -\rho w, \qquad h_2 = \rho v, \qquad h_3 = -(u + c_1) \tag{3.5}$$

These expressions have the remarkable property of satisfying the initial Cauchy condition (2.11), and they can easily be shown to be a solution of the system (2.13) because of the relations (1.1) and (1.2).

Substituting (3.5) into (3.1) to (3.4), we obtain the following system of equations for the determination of the extremal characteristic surface of the first family: (3.6)

$$A = a\sqrt{1 + \varphi_y^2 + \varphi_z^2} = -u + v\varphi_y + w\varphi_z$$

$$\frac{A}{a^2}\,\frac{\varphi_y}{v} + \frac{1}{u + c_1}, \qquad \frac{A}{a^2} = \frac{\varphi_z}{w} + \frac{1}{u + c_1}, \qquad \frac{\partial v\rho\,(u + c_1)}{\partial y} + \frac{\partial w\rho\,(u + c_1)}{\partial z} = 0$$

Let us replace the unknown functions v and w on Σ by ω and ε by means of Formulas

$$v = \omega\cos\varepsilon, \qquad w = \omega\sin\varepsilon, \qquad v^2 + w^2 = \omega^2$$

Eliminating A and taking into account the fact that $\rho = \rho\,(u^2 + \omega^2)$ and $a = a\,(u^2 + \omega^2)$, we can transform the system (3.6) into

$$u + c_1 = -\omega \frac{a}{\sqrt{u^2 + \omega^2 - a^2}} \tag{3.7}$$

$$\varphi_y = \omega\cos\varepsilon\, \frac{2u + c_1}{\omega^2 - u\,(u + c_1)} \qquad \varphi_z = \omega\sin\varepsilon\, \frac{2u + c_1}{\omega^2 - u\,(u + c_1)} \tag{3.8}$$

$$\frac{\partial \cos\varepsilon\,(u + c_1)\,\omega\rho}{\partial y} + \frac{\partial \sin\varepsilon\,(u + c_1)\,\omega\rho}{\partial z} = 0 \tag{3.9}$$

This system yields the functions $u(y,z)$, $\varepsilon(y,z)$, $\omega(y,z)$ and $\varphi(y,z)$ on Σ. Let us analyze the systems (3.7) – (3.9).

The final relation (3.7) shows that in the space of the velocity hodograph the surface Σ is representable as an axially symmetric surface with u as axis of symmetry. Thus, the relation (3.7) permits one to consider u on Σ as a known function of ω, i.e. $u = u(\omega)$.

In the determination of Σ it is necessary to satisfy the boundary conditions. Firstly, Σ must pass through the given contour Γ. This means that in the yz plane there is given the contour γ and the values of φ on it. Secondly, the surface Σ must pass through some contour L which belongs to the given characteristic surface Σ_1. This means that on some contour l of the yz plane the relations (3.7) to (3.9) must be satisfied by the given values of the gas-dynamic function because of flow continuity. In the sequel we shall solve the inverse problem: we will select on Σ_1 some contour L satisfying the relations (3.7) to (3.9), and by means of (3.7) to (3.9) we will construct the surface Σ passing through the contour L. After that we construct the contour Γ on Σ by means of the values of $\psi = \psi(\chi)$ given on Γ_1. The contour Γ will correspond to the selected contour L.

Let us choose an arbitrary point on Σ_1. The relation (3.7) permits us to determine at once the contour L on Σ_1. This determines also the contour l on the yz plane and the values of φ on l. With these data one can evaluate the derivative $d\varphi/ds$ on l, where s is the arc length of l. On the other hand, the relations (3.8) determine φ_y and φ_z on l, and, hence, also the derivative $d\varphi/ds$. It is obvious that in the general case the values $d\varphi/ds$ on l, evaluated by the first and second method, will not coincide. This indicates that the problem has no solution in general.

However, the problem can be solved if one assumes that there can occur a break of the surface σ on the contour Γ_1

In this case an infinite number of characteristic surfaces Σ_{11} may emerge from the contour Γ_1. Each Σ_{11} is determined only by the given surface Σ_1 and by an arbitrary function $\delta_1(\Gamma_1)$ chosen along Γ_1 (the spatial analog of the Prandtl-Meyer flow). For the function $\delta_1(\Gamma_1)$ one may take, for example a dihedral angle between two tangent planes to the surfaces Σ_1 and Σ_{11} at points of the contour Γ_1. For an arbitrary chosen point of Σ_1, we select a function $\delta_1(\Gamma_1)$ and thereby an initial characteristic surface Σ_{11}, such that on the constructed contour l the values of $d\varphi/ds$, evaluated by the first and second method coincide. In this manner one constructs the required contour L and determines the initial conditions for the solution of the system (3.7) – (3.9).

The system of equations (3.7) to (3.9) can be reduced to a system of a known type.

Let us introduce a new function V_0 by means of Formula

$$V_0 = \omega \frac{2u + c_1}{\omega^2 - u(u + c_1)} \tag{3.10}$$

Since by (3.7), $u = u(\omega)$, the relation (3.10) can be considered as an implicit determination of $\omega = \omega(V_0)$. This permits one to consider Expression

$$\rho_0(V_0) = \rho \frac{\omega(u + c_1)}{V_0} \tag{3.11}$$

as a function of V_0.

Let us set

$$u_0 = \varphi_y, \quad v_0 = \varphi_z, \quad x_0 = y, \quad y_0 = z \qquad (V_0^2 = u_0^2 + v_0^2) \tag{3.12}$$

Equating the cross derivatives of Expression (3.8), we may rewrite the system (3.8) - (3.9) in the form

$$\frac{\partial u_0}{\partial y_0} - \frac{\partial v_0}{\partial x_0} = 0, \qquad \frac{\partial \rho_0 u_0}{\partial x_0} + \frac{\partial \rho_0 v_0}{\partial y_0} = 0 \qquad (3.13)$$

The system (3.13) describes plane irrotational "flows" of a compressible fluid with a "potential" φ which is of the form of an extremal characteristic of the surface Σ. To continue the analogy, the "velocity of sound" for this "flow" is computed by means of Formula

$$a_0^2 = -V_0\rho_0 / \rho_0'$$

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